A SLIGHTLY TWISTED RADIAL-SLOT JET DISCHARGING FROM AN ANNULAR SOURCE OF FINITE DIAMETER

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An approximate solution is presented for the problem of laminar and turbulent slightly twisted radial-slot jets of an incompressible liquid, discharging from an annular source of finite diameter.

In all solutions for the problem of laminar and turbulent slightly twisted radial-slot jets ([1, 2], etc.), it was assumed that the radius of the annular source is zero (Fig. 1a). Consequently, the above-enumerated solutions satisfactorily describe the actual flow only at comparatively great distances from the nozzle, exceeding the nozzle radius several times over.

Ginevskii [3] solved the problem of the discharge of laminar and turbulent untwisted jets from an annular source of finite radius.

Below we will attempt to extend the Ginevskii results to slightly twisted jet flows, as shown in Fig. 1.

We should note that for the flows shown in Fig. 1b, as the radius of the annular source tends to zero, the resulting solution changes into an asymptotic solution that is quite close, in terms of numerical values, to the above-cited solutions [1, 2]. It is natural that this solution, derived for the problem (Fig. 1c), is valid in a region that is not too close to the axis of symmetry where the jets collide, where pressure gradients are developed, and where the flow apparently becomes unstable.

The boundary-layer equations for an isobaric slightly twisted jet are written in cylindrical coordinates in the form

$$u\frac{du}{dx}+v\frac{du}{dy}=\frac{1}{\rho}\frac{d\tau_1}{dy}, \tag{1}$$

$$u\frac{dw}{dx}+v\frac{dw}{dy}+\frac{uw}{x}=\frac{1}{\rho}\frac{d\tau_2}{dy}, \tag{2}$$

$$\frac{d(ux)}{dx}+\frac{d(vx)}{dy}=0. \tag{3}$$

Having integrated (1) and (2) over the boundary layer of finite width and having eliminated the velocity v by means of (3), we find the invariance of the momentum of the jet in the radial direction and of the angular momentum of the jet in the circumferential direction

$$I=2\pi\rho x\int_{-\delta_1}^{\delta_1}u^2(y)\,dy=\text{const}, \tag{4}$$

$$M=2\pi\rho x^2\int_{-\Delta}^{\Delta}u(y)\,w(y)\,dy=\text{const}. \tag{5}$$

Having presented the profiles for the friction-stress components across the jet in the form of polynomials of degree y and having determined the coefficients of these polynomials from the boundary conditions which follow from the differential equations of motion (1)-(3), we have

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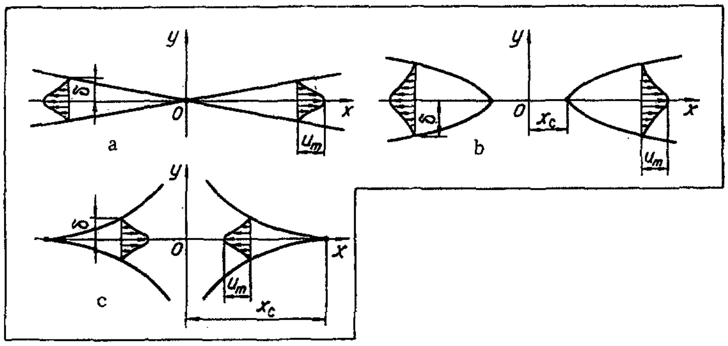

Fig. 1. Flow diagram: a) point source; b) external radial-slot jet; c) internal radial-slot jet.

$$\tau_1 = \rho u_m u_m' \delta_1 y_1 (1 - y_1)^2, \tag{6}$$

$$\tau_2 = \rho u_m \left(w_m' + \frac{w_m}{x} \right) \delta_2 y_2 (1 - y_2)^2. \tag{7}$$

Relations (6) and (7) are identically valid for both the laminar and the turbulent jet.

In conjunction with the corresponding formulas for the components of the shearing stress and with the integral relations (4) and (5), Eqs. (6) and (7) make it possible to close the problem completely.

We note that (1) is independent of (2) and corresponds in accuracy to the case of flow in a jet without twisting [3]. The same applies to (4) and (6). We will therefore use all of the results for the radial direction in the following, without any further reference to [3].

In the case of laminar flow in the jet

$$\tau_1 = \mu du/dy, \ \tau_2 = \mu dv/dy. \tag{8}$$

This profile for the velocity u has the form

$$\frac{u}{u_m} = 1 - 6y_1^2 + 8y_1^3 - 3y_1^4. \tag{9}$$

Changes in the axial velocity u_m and in the boundary-layer thickness δ_1 are subject to the following quantitative relationships:

$$u_m = \frac{0.186}{x_j} \left(\frac{K_1^2}{\nu} \right)^{1/3} (\pm x_0^3 \mp 1)^{-1/3}, \tag{10}$$

$$\frac{\delta_1}{x_j} = 8.025 \left(\frac{\nu^2}{K_1} \right)^{1/3} (\pm x_0^3 \mp 1)^{2/3} / x_0. \tag{11}$$

The upper signs correspond to the case of an external-slot jet (see Fig. 1b); the lower signs correspond to the case of an internal radial-slot jet (see Fig. 1 c), and for the first we have $x_0 \geq 1$, while for the second we have $1 \geq x_0 > 0$.

Substituting (8) into (7) and integrating for the conditions $w = w_m$ when $y = 0$ and $w = 0$ when $y = \delta_2$, we have

$$\frac{w}{w_m} = 1 - 6y_1^2 + 8y_1^3 - 3y_1^4, \tag{12}$$

$$w_m + \frac{1}{12\nu} u_m \left(w_m' + \frac{w_m}{x} \right) \delta_2^2 = 0. \tag{13}$$

It follows from a comparison of (9) and (12) that the velocity profiles in the direction of the x-axis and the circumferential direction are identical.

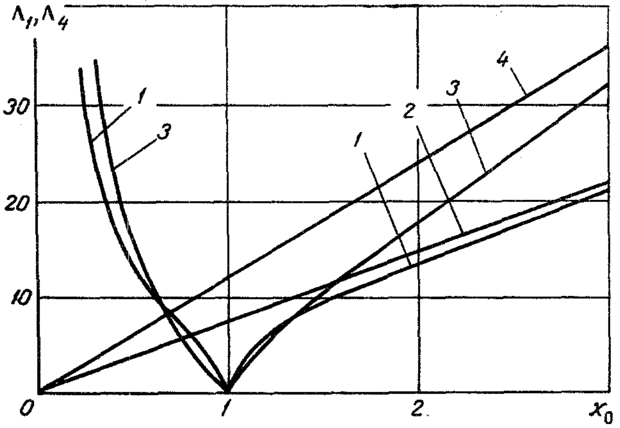

Fig. 2. Jet thickness. For laminar flow curve 1 has been calculated from (11) and (17); curve 2 has been calculated from (20). For turbulent flow curve 3 has been calculated from (26) and (28), and curve 4 has been calculated from (30).

Equation (13) enables us to determine the change in the jet parameters in the circumferential direction. From the integral condition (5), (9), and (12) we have

$$\delta_2 = K_2/4\pi\gamma(\varepsilon)\, x u_m x w_m, \tag{14}$$

$$\gamma(\varepsilon) = \begin{cases} (2/5) - (8/35)\varepsilon^{-2} + (1/7)\varepsilon^{-3} - (1/35)\varepsilon^{-4} & (\varepsilon \geqslant 1), \\ \varepsilon\left[(2/5) - (8/35)\varepsilon^{2} + (1/7)\varepsilon^{3} - (1/35)\varepsilon^{4}\right] & (\varepsilon \leqslant 1). \end{cases} \tag{15}$$

After substitution of (14) into (13) and after integration in limits from x_j to x and from $w_m = \infty$ to w_m in the assumption that $\varepsilon = \delta_1/\delta_2 = \text{const}$ and, consequently, that $\gamma(\varepsilon) = \text{const}$, we find

$$w_m = \frac{0.0532}{\gamma(\varepsilon)\, x_j^2}\left(\frac{K_2^2}{\nu}\right)^{1/2}\left(\frac{\nu}{K_1^2}\right)^{1/6} x_0^{-1}(\pm x_0^3 \mp 1)^{-1/3}. \tag{16}$$

After substitution of (16) into (14) we have

$$\frac{\delta_2}{x_j} = 8.025\left(\frac{\nu^2}{K_1}\right)^{1/3}(\pm x_0^3 \mp 1)^{2/3}/x_0. \tag{17}$$

From (11) and (17) we see that

$$\varepsilon = \delta_1/\delta_2 = \text{const} = 1, \tag{18}$$

which was what had been assumed prior to the integration of (16).

When $x_0 \gg 1$ (this condition is equivalent to the case $x_j \to 0$), formulas (10), (11), (16), and (17) are markedly simplified:

$$u_m = 0.186\left(\frac{K_1^2}{\nu}\right)^{1/3}\frac{1}{x}, \tag{19}$$

$$\delta_1 = \delta_2 = 8.025\left(\frac{\nu^2}{K_1}\right)^{1/3} x, \tag{20}$$

$$w_m = 0.186\left(\frac{K_2^2}{\nu}\right)^{1/2}\left(\frac{\nu}{K_1^2}\right)^{1/6}\frac{1}{x^2}. \tag{21}$$

The corresponding formulas of the solutions cited in [2], in our notation, have the form

$$u_m = 0.193\left(\frac{K_1^2}{\nu}\right)^{1/3}\frac{1}{x}, \tag{22}$$

$$w_m = 0.193\left(\frac{K_2^2}{\nu}\right)^{1/2}\left(\frac{\nu}{K_1^2}\right)^{1/6}\frac{1}{x^2} \tag{23}$$

and differ from (19) and (21) only in their numerical coefficients.

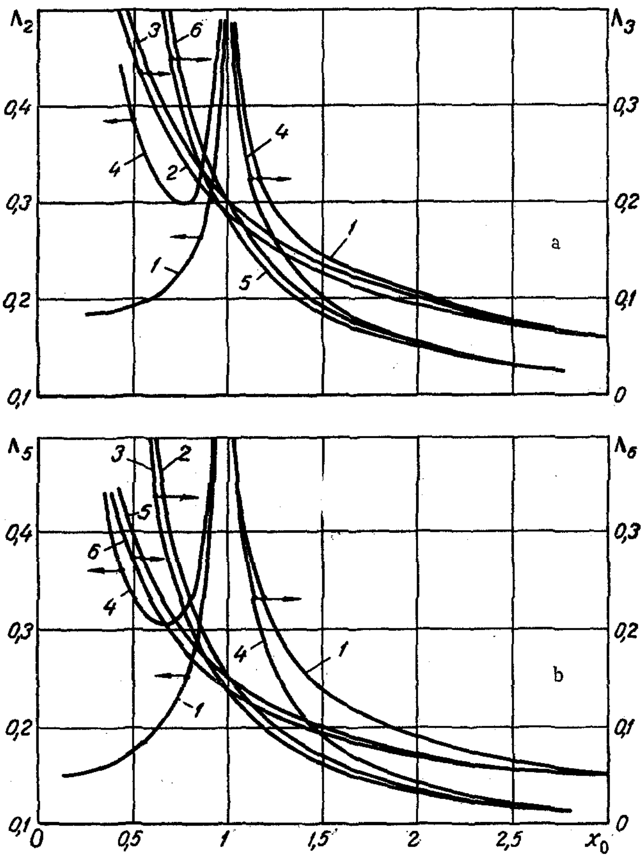

Fig. 3. Change in the velocities u_m and w_m for laminar flow (a) and for turbulent flow (b): a: 1) according to (10); 2) according to (19); 3) according to (22); 4) according to (16); 5) according to (21); 6) according to (23); b: 1) according to (25); 2) according to (29); 3) according to (32); 4) according to (27); 5) according to (31); 6) according to (33).

Figure 2 shows the functions

$$\Lambda_1 = (\delta_1/x_j)(K_1/\nu^2)^{1/3} = (\delta_2/x_j)(K_1/\nu^2)^{1/3},$$

calculated from (11), (17), and (20).

Figure 3a shows the graphs of the functions

$$\Lambda_2 = u_m x_j \left(\frac{\nu}{K_1^2}\right)^{1/3}; \quad \Lambda_3 = w_m x_j \left(\frac{\nu}{K_2^2}\right)^{1/2} \left(\frac{K_1^2}{\nu}\right)^{1/6},$$

calculated from (10), (16), (19), and (21-23).

It follows from Fig. 3a that when $x_0 > 2.5$ the calculations with all of the formulas are close to each other.

However, the difference in the results of the approximate and exact solutions for a slightly twisted radial-slot jet discharging from a point source (formulas (19), (21)-(23)) is not great and falls within the limits of conventional differences between the theory for a layer of finite thickness and the theory for an asymptotic layer.

In the case of turbulent flow in a jet with slight twisting we can use the familiar formula for the Reynolds stresses

$$\tau_1 = \rho \varkappa \delta_1 u_m \, \partial u/\partial y, \quad \tau_2 = \rho \varkappa \delta_2 u_m \, \partial v/\partial y. \tag{24}$$

Substituting (24) into (6) and (7), and carrying out the transformations similar to those which were used above for the laminar jet, we derive (9) and (12) from [3] for the velocity profiles:

$$u_m = \frac{0.152}{x_j}\left(\frac{K_1}{\varkappa}\right)^{1/2} (\pm x_0^2 \mp 1)^{-1/2}, \tag{25}$$

$$\frac{\delta_1}{x_j} = 12\varkappa\,(\pm x_0 \mp x_0^{-1}), \tag{26}$$

and instead of (16), integrating within the same limit from x_j to x and from $w_m = \infty$ to w_m, retaining the earlier assumption that $\gamma(\varepsilon) = \text{const}$, we find

$$w_m = \frac{0.0435}{\gamma(\varepsilon)\, x_j^2}\left(\frac{K_2}{\varkappa}\right)\left(\frac{\varkappa}{K_1}\right)^{1/2} x_0^{-1} (\pm x_0^2 \mp 1)^{-1/2}, \tag{27}$$

and it thus follows from (12) that

$$\frac{\delta_2}{x_j} = 12\varkappa\,(\pm x_0 \mp x_0^{-1}). \tag{28}$$

From (26) and (28) we have the earlier result shown in (18).

We note that since the jet is slightly twisted, according to [3], we can assume that $\varkappa = 0.0125$.

From (25)-(28) when $x_0 \gg 1$ (or, what is the same, $x_j \to 0$) we have

$$u_m = 0.152\left(\frac{K_1}{\varkappa}\right)^{1/2} \frac{1}{x}, \tag{29}$$

$$\delta_1 = \delta_2 = 12\varkappa x, \tag{30}$$

$$w_m = 0.152\left(\frac{K_2}{\varkappa}\right)\left(\frac{\varkappa}{K_1}\right)^{1/2} \frac{1}{x^2}. \tag{31}$$

The corresponding formulas for the solution given in [2] for u_m and w_m have the form

$$u_m = 0.141\left(\frac{K_1}{\varkappa}\right)^{1/2} \frac{1}{x}, \tag{32}$$

$$w_m = 0.141\left(\frac{K_2}{\varkappa}\right)\left(\frac{\varkappa}{K_1}\right)^{1/2} \frac{1}{x^2}, \tag{33}$$

and they differ from (29) and (30) only in the values of the numerical coefficients.

Figures 2 and 3b show the changes in the following quantities:

$$\Lambda_4 = \frac{\delta_1}{\varkappa x_j} = \frac{\delta_2}{\varkappa x_j}, \quad \Lambda_5 = u_m x_j\left(\frac{\varkappa}{K_1}\right)^{1/2},$$

$$\Lambda_6 = w_m x_j\left(\frac{\varkappa}{K_2}\right)\left(\frac{K_1}{\varkappa}\right)^{1/2}.$$

In conclusion, we know that it follows from Fig. 3a and b and from (16) and (27) that for an internal radial-slot jet the twisting velocity w_m varies from ∞ for $x_0 = 1$ to some minimum value of w_m^* for $x_0 = x_0^*$, and it then increases to ∞ when $x_0 = 0$.

It follows from (16) for the laminar regime that

$$x_0^* = 2^{-1/3}, \quad w_m^* = 0.296\,(K_2^2/\nu)^{1/2}\,(\nu/K_1^2)^{1/6}.$$

For the turbulent regime we correspondingly find from (27) that

$$x_0^* = 2^{-1/2}, \quad w_m^* = 0.304\,(K_2/\varkappa)\,(\varkappa/K_1)^{1/2}.$$

NOTATION

I	is the jet momentum;
$K_1 = I/\rho$	is the kinematic momentum of the jet;

$K_2 = M/\rho$	is the kinematic angular momentum of the jet;
M	is the angular momentum of the jet;
u, w, and v	are the velocity components along the x-axis, in the circumferential direction and along the y-axis, respectively;
$u_m = u(0, x)$;	
$w_m = w(0, x)$;	
$u'_m = du_m/d_x$;	
$w'_m = dw_m/dx$;	
x_j	is the radius of the annular source;
$x_0 = x/x_j$;	
$y_i = y/\delta_i$ $(i = 1, 2)$;	
δ_1, δ_2	are the half-thicknesses of the jet in the direction of the x-axis and in the circumferential direction, respectively
$\Delta = \min(\delta_1, \delta_2)$;	
$\varepsilon = \delta_1/\delta_2$	
$\varkappa$	is the experimentally determined constant;
μ	is the dynamic viscosity of the fluid;
$\nu = \mu/\rho$	is the coefficient of kinematic viscosity for the fluid;
ρ	is the density;
τ_1 and τ_2	are the shearing-stress components along the x-axis and in the circumferential direction.

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